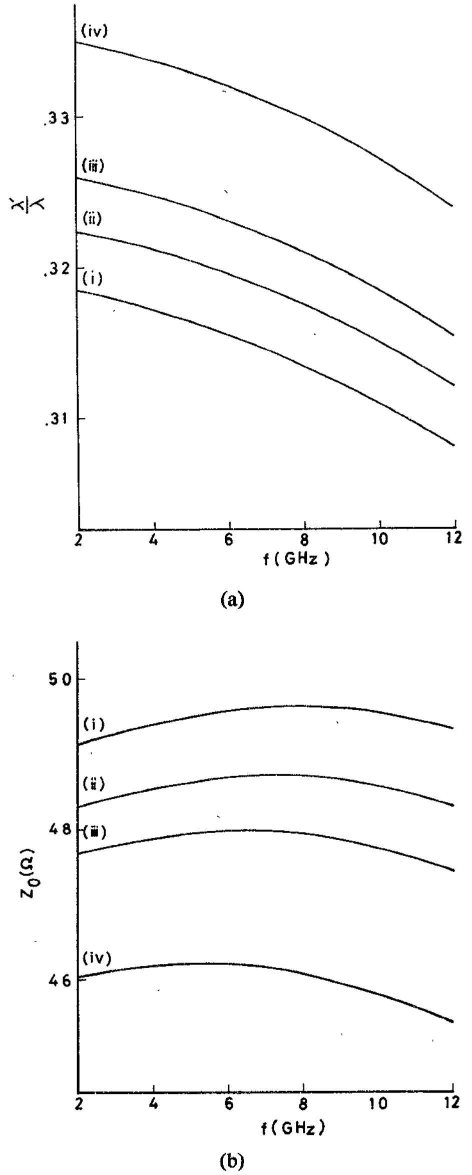

Fig. 3. (a) Normalized guide wavelength. (b) Characteristic impedance. $\varepsilon_r = 20$; $h = 1.0$ mm; $A = 0.2$ mm; $b = 0.7$ mm; $W = b - a$ (i) $t/W = 0.00$; (ii) $t/W = 0.02$; (iii) $t/W = 0.04$; (iv) $t/W = 0.10$.

These numerical calculation were carried out by the electronic computer FACOM 230-75. The calculation time was about 6 s for the wavelength and 0.5 s for the characteristic impedance.

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Analytical IC Metal-Line Capacitance Formulas

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***Abstract*—In semiconductor IC technology, capacitances formed by the multilevel interconnection metal lines usually dominate circuit performance. However, for lack of accurate formulas, a numerical method usually has to be used to determine these capacitances. Two analytical capacitance formulas were derived using approximate conformal mapping techniques. One formula gives the capacitance of a finite-thickness metal line over a conducting ground plane, or over a silicon surface. The other formula gives the capacitance of the same metal line, but with an additional conducting metal line over it. The formulas are most accurate for metal lines whose width exceeds the dielectric thickness; accuracy increases with linewidth. They are accurate to 1 percent for a metal line whose width is comparable to the dielectric thickness. With these simple formulas, statistical distribution of the metal-line capacitances can be easily determined in a few seconds of computer time.**

I. Introduction

In semiconductor integrated-circuit technology, capacitances formed by the metal interconnection lines usually dominate the circuit performance of the chip. It is therefore important for a circuit designer to determine metal-line capacitances accurately. Since the basic structure is similar to that of a microwave stripline, the structure has been well analyzed [1]–[8]. However, most of the treatment is either limited to a metal line of infinitesimal thickness, or a complicated numerical method is used to obtain the capacitances. The numerical method requires a computer and long computation time. Using conformal mapping techniques, accurate analytical capacitance formulas have been derived for a single rectangular metal line. Two analytical capacitance formulas are given: one gives the capacitance of a rectangular metal line over a conducting ground plane; the other gives the capacitance of the same metal line with two conducting ground planes, one above it and the other below it.

II. A Metal Line Over a Ground Plane

Fig. 1 shows a rectangular metal line above a ground plane. The metal has a width W, thickness t, and is separated from the ground plane by the distance h. The ground plane may be a silicon surface operating in an accumulation region. Because of the symmetry of the structure, only one-half of the structure needs to be considered. The half metal line $BCDE$ is extended to infinity at A and F in Fig. 2(a) to apply conformal mapping techniques and get a simpler formula. Boundary conditions are that the metal be at unity voltage, and ground plane be at 0 voltage, and the normal electric field vanishes along the lines MB and EKG. The Schwarz–Christoffel transformation (2)

$$\frac{\pi z}{h} = \frac{p+1}{p^{1/2}} \tanh^{-1} R + \left(\frac{p-1}{p^{1/2}}\right)\left(\frac{R}{1-R^2}\right) + \ln\left(\frac{Rp^{1/2}-1}{Rp^{1/2}+1}\right) \quad (1)$$

$$R = \left(\frac{w+1}{w+p}\right)^{1/2} \quad (2)$$

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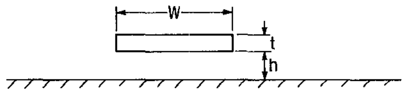

Fig. 1. A metal line above a ground plane.

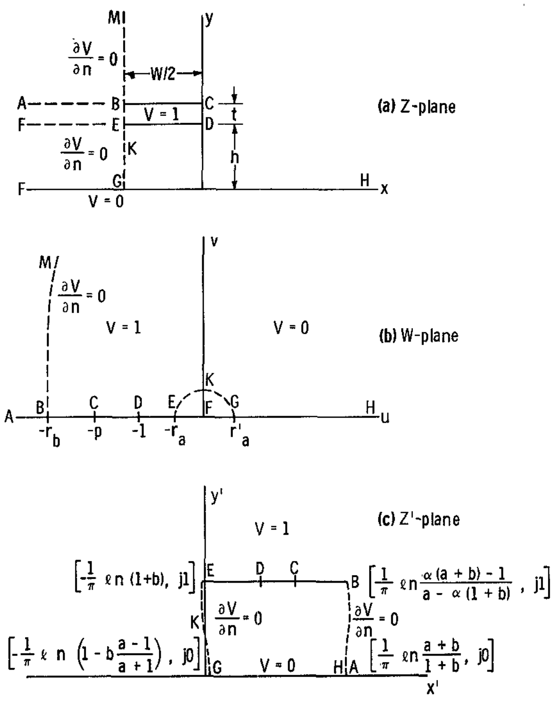

Fig. 2. Transformations for Fig. 1.

$$p = 2B^2 - 1 + \sqrt{(2B^2 - 1)^2 - 1} \tag{3}$$

$$B = 1 + t/h \tag{4}$$

transforms the Z plane in Fig. 2(a) into the W plane in Fig. 2(b). Another transformation

$$z' = 1/\pi \ln \left[\frac{d + w(a + b)}{ad + w(1 + b)}\right] \tag{5}$$

$$d = 2(1/r_a + 1/r_a')^{-1} \tag{6}$$

$$b = \frac{r_a' - r_a}{r_a' + r_a} \tag{7}$$

$$a = \alpha(1 + b^2) - b + \sqrt{(1 - b^2)[\alpha^2 - (1 - b\alpha)^2]} \tag{8}$$

$$\alpha = r_b/d$$

transforms the region to the right of $MBCDEKGH$ in the W plane into a region interior to and bounded by an almost-parallel plate in the Z plane.

Since the structure is an almost-parallel plate, the capacitance can be approximated by that of a parallel plate as

$$C/\varepsilon \approx \overline{BE} = 1/\pi \ln \frac{\alpha(a + b) - 1}{a - \alpha(1 + b)}$$

$$\approx 1/\pi \ln [r_b/r_a + \sqrt{(r_b/r_a)^2 - 1}] \tag{9}$$

$$\approx 1/\pi \ln (2r_b/r_a) \tag{10}$$

TABLE I
COMPARISON BETWEEN THE CAPACITANCES CALCULATED BY (24) AND BY THE NUMERICAL METHOD [7]

W/h	t/h	$C_1=C/\varepsilon$ (formula)	$C_2=C/\varepsilon$ (numerical)	$(C_1-C_2)/C_2$
1.12	0.318	3.552	3.591	-1.1%
2.01	0.485	4.762	4.793	-0.6%
2.52	0.318	5.227	5.242	-0.3%
2.72	0.802	5.789	5.815	-0.4%
3.18	0.936	6.371	6.394	-0.4%
3.63	0.802	6.806	6.820	-0.2%
3.68	0.485	6.662	6.661	<0.05%
3.79	1.116	7.136	7.159	-0.3%
4.24	0.936	7.542	7.544	<0.05%
5.44	1.200	8.961	8.978	-0.2%
6.36	0.802	9.760	9.754	+0.1%
7.42	0.936	10.955	10.935	+0.2%
8.21	1.805	12.122	12.140	-0.1%
9.85	1.805	13.842	13.828	+0.1%
11.90	1.453	15.855	15.847	+0.1%
14.78	1.805	18.951	18.974	+0.1%
22.22	4.070	27.074	27.077	<0.05%
58.25	7.113	63.940	63.943	<0.05%

where ε is the permittivity of the dielectric. Equation (9) is obtained by assuming that $r_a = r_a'$ $(b = 0)$ and (10) for $r_b \gg r_a$. The total capacitance of a rectangular metal line over a conducting ground plane is twice that of (10), so

$$C/\varepsilon = \frac{2}{\pi} \ln 2r_b/r_a. \tag{11}$$

Since r_a is much smaller and r_b is much larger than 1, r_a and r_b can be approximated by the iteration method.

$$\ln r_a \approx -1 - \frac{\pi W}{2h} - \frac{p + 1}{p^{1/2}} \tanh^{-1} (p^{-1/2}) - \ln \frac{p - 1}{4p} \tag{12}$$

$$r_b \approx \eta + \frac{p + 1}{2p^{1/2}} \ln \Delta \tag{13}$$

$$\eta = p^{1/2} \left\{\frac{\pi W}{2h} + \frac{p + 1}{2p^{1/2}} \left[1 + \ln \frac{4}{p - 1}\right] - 2 \tanh^{-1} p^{-1/2}\right\} \tag{14}$$

$$\Delta = \text{larger value, } \eta \text{ or } p. \tag{15}$$

Based on (11)–(15) the capacitances are obtained and compared with the results obtained by the numerical method (7). The errors are within 1 percent if $W/h \gtrsim 1$. The comparison is listed in Table I. The formula is more accurate for a thicker metal line.

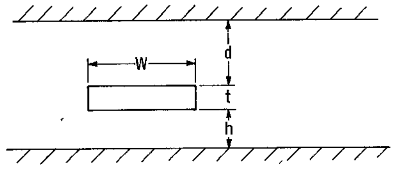

Fig. 3. Rectangular metal line with two conducting ground planes.

III. Metal Line with Two Ground Planes

Fig. 3 shows a rectangular metal line with two conducting ground planes, one above it and the other below it. The illustration shows an approximate structure for a much wider second metal line lying above the first metal line. The metal line is W in width, t in thickness, and is separated from the ground planes by distances h and d. One-half of Fig. 3 is reproduced in Fig. 4(a). The boundary conditions are that the ground planes be at 0 V, the metal be at unity voltage, and the normal electric field be vanished along lines MPB and EQG. To apply conformal mapping techniques and get a simpler formula, the metal is extended to A and F at $-\infty$. The differential form of the Schwarz–Christoffel transformation is

$$\frac{dz}{dw} = A\,\frac{(w+1)^{1/2}(w+p)^{1/2}}{w(w-q)}. \quad (16)$$

After integration is carried out, the following boundary conditions

$$\begin{aligned} z &= j(t+h), & w &= -p \\ z &= jh, & w &= -1 \\ z &= -\infty, & w &= 0 \\ z &= \infty, & w &= q \end{aligned} \quad (17)$$

are substituted to obtain the transformation,

$$\frac{\pi z}{2h} = \alpha \tanh^{-1}\sqrt{\frac{(p+q)(w+1)}{(1+q)(w+p)}} - \gamma \tanh^{-1}\sqrt{\frac{w+1}{w+p}} - \tanh^{-1}\sqrt{\frac{w+p}{p(w+1)}} \quad (18)$$

where

$$\alpha = \frac{h+d+t}{h}$$

$$\gamma = \frac{d}{h}$$

$$p = \frac{q^2}{\gamma^2}$$

$$q = \tfrac{1}{2}\left[\alpha^2 - \gamma^2 - 1 + \sqrt{(\alpha^2-\gamma^2-1)^2 - 4\gamma^2}\right]. \quad (19)$$

The results can be checked by differentiating (18). The capacitance of the structures shown in Fig. 4(b) is easily found from Fig. 4(c) to be

$$C/\varepsilon = 1/\pi \ln R_B/R_A. \quad (20)$$

Since the capacitance of the structure shown in Fig. 3 is twice that of Fig. 4, the total capacitance of a rectangular metal line with two conducting ground planes is

$$C/\varepsilon = 2/\pi \ln R_B/R_A. \quad (21)$$

The parameters R_A and R_B can be obtained approximately by Taylor series expansion. The results are

$$\ln R_A \approx -\pi W/2h - 2\alpha \tanh^{-1}\sqrt{(p+q)[p(1+q)]} + 2\gamma \tanh^{-1} p^{-1/2} + \ln\,[4p/(p-1)] \quad (22)$$

$$\ln R_B \approx \gamma^{-1}\{\pi W/2h + 2\alpha \tanh^{-1}\sqrt{(1+q)/(p+q)} + \gamma \ln\,[(p-1)/4] - 2\tanh^{-1} p^{-1/2}\}. \quad (23)$$

The capacitance calculated from (21)–(23) is compared with that calculated by numerical method [7], as shown in Table II. The errors are within 1 percent for $W/h \geq 0.5$, and $d/h \geq 0.5$.

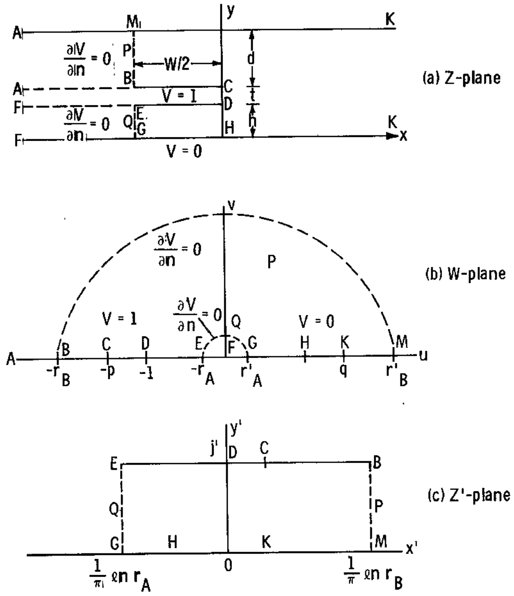

Fig. 4. Transformations for Fig. 3.

TABLE II
Comparison Between the Capacitances Calculated by (25) and by the Numerical Method [7]

W/h	t/h	d/h	W/d	C/ε (formula)	C/ε (numerical)	% difference
2.765	0.668	2.838	0.974	8.052	8.036	+0.2%
2.725	0.795	3.233	0.843	7.616	7.611	+0.1%
2.725	0.795	4.119	0.662	6.792	6.777	+0.2%
2.725	0.795	5.813	0.469	6.285	6.255	+0.5%
2.725	0.795	6.698	0.407	6.183	6.139	+0.7%
5.069	0.668	2.838	1.786	12.325	12.279	+0.4%
10.490	3.059	23.147	0.453	15.140	15.059	+0.5%
10.490	3.059	19.738	0.531	15.213	15.175	+0.3%
10.490	3.059	13.217	0.794	15.619	15.638	-0.1%
10.490	3.059	9.808	1.070	16.316	16.333	-0.1%
58.252	7.152	37.573	1.012	65.169	65.102	+0.1%

IV. Summary

The capacitance of a rectangular metal line over a conducting ground plane, for $W/h \gtrsim 1$, is

$$C = 2\varepsilon/\pi \ln 2R_b/R_a$$

$$\ln R_a = -1 - \pi W/2h - [(p+1)/p^{1/2}] \tanh^{-1} p^{-1/2} - \ln [(p-1)/4p]$$

$$R_b = \eta + [(p+1)/2p^{1/2}] \ln \Delta$$

$$\eta = p^{1/2}\{\pi W/2h + [(p+1)/2p^{1/2}][1 + \ln 4/(p-1)] - 2 \tanh^{-1} p^{-1/2}\}$$

$$\Delta = \text{larger value, } \eta \text{ or } p$$

$$p = 2B^2 - 1 + \sqrt{(2B^2-1)^2 - 1}$$

$$B = 1 + t/h. \quad (24)$$

The accuracy of (24) is about 1 percent when $W/h \gtrsim 1$.

The capacitance of a rectangular metal line with two conducting ground planes, for $W/h \geq 0.5$, $d/h \geq 0.5$, is

$$C/\varepsilon = 2/\pi \ln R_B/R_A$$

$$\ln R_A = -\pi W/2h - 2\alpha \tanh^{-1} \sqrt{(p+q)/p(1+q)} + 2\gamma \tanh^{-1} p^{-1/2} + \ln 4p/(p-1)$$

$$\ln R_B = \gamma^{-1}\{\pi W/2h + 2\alpha \tanh^{-1} \sqrt{(1+q)/(p+q)} + \gamma \ln (p-1)/4 - 2 \tanh^{-1} p^{-1/2}\}$$

$$\alpha = (h+d+t)/h$$

$$\gamma = d/h$$

$$p = q^2/\gamma^2$$

$$q = \tfrac{1}{2}[\alpha^2 - \gamma^2 - 1 + \sqrt{(\alpha^2-\gamma^2-1)^2 - 4\gamma^2}]. \quad (25)$$

The errors are less than 1 percent. Both formulas are more accurate for a wider and/or a thicker metal line. Once the capacitance is found, the characteristic impedance of the composite transmission line is easily obtained. Aside from a constant, the characteristic impedance is the inverse of the capacitance.

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Equivalent Circuit of a Narrow Axial Strip in Waveguide

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***Abstract*—A theoretical determination is made of the two-port equivalent circuit of a narrow strip located axially in a rectangular waveguide such that it extends partially or completely across the waveguide narrow dimension. The analysis is based upon derivation of a variational expression for a field quantity from which can be determined the reflection coefficient and the equivalent-circuit parameters. Experimental input susceptance values agree closely with the theory. The analysis shows that the T-equivalent network of a nontouching strip has a series-resonant shunt circuit. This element has application in filter and impedance-transforming networks, in planar circuits, and in fin-line structures.**

I. Introduction

This short paper presents a theoretical determination, with experimental verification, of the equivalent circuit of a narrow infinitesimally thin perfectly conducting strip which is partially or completely inserted in a rectangular waveguide in such a manner that the strip surface is parallel to the narrow waveguide wall.

Konishi *et al.* [1]–[3] have developed a method for the design of planar circuits, by which the circuit elements are located on a metal sheet which is inserted axially into a waveguide; its advantages include low cost and ease of mass production. Meier [4]–[7] has advocated fin line, in which metal fins printed on a dielectric substrate bridge the broad walls of a rectangular waveguide, as a propagating structure for millimeter-wave integrated circuits. The geometry considered in this short paper belongs to the general form defined by those papers. Although the analysis presented here is restricted to narrow strips, it is applicable to the design of bandpass filters, diode mounts, and tuning elements of the form described by Konishi *et al.* [2], [3].

The narrow axial nontouching thin strip has not previously been subjected to theoretical analysis. Konishi *et al.* [2], [3] used a Rayleigh–Ritz variational technique to obtain an equivalent circuit for a uniform strip which extends across the entire waveguide height. Their method requires knowledge of the modes in the two sections of the waveguide bifurcated by the strip; thus it is not readily applicable to the nontouching strip.

The approach used here is based upon the variational method used previously by the present authors for the analysis of a thin

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